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Saddlepoints in P-Pivot Classes of Skew Matrices*

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(September 13, 1974)

A finite class of skew matrices can be associated via "principal pivots" with a self-dual linear system. Given a row index h, it is known that there is a skew matrix in the class with nonnegative hth row. Using this "saddlepoint theorem," we prove a similar result for nonpositive rows. An open question is whether such a class of skew matrices contains one with both a nonnegative row and a nonpositive row. We show that this "double saddlepoint property" holds for arbitrarily large degenerate cases and for all skew matrices of small order.

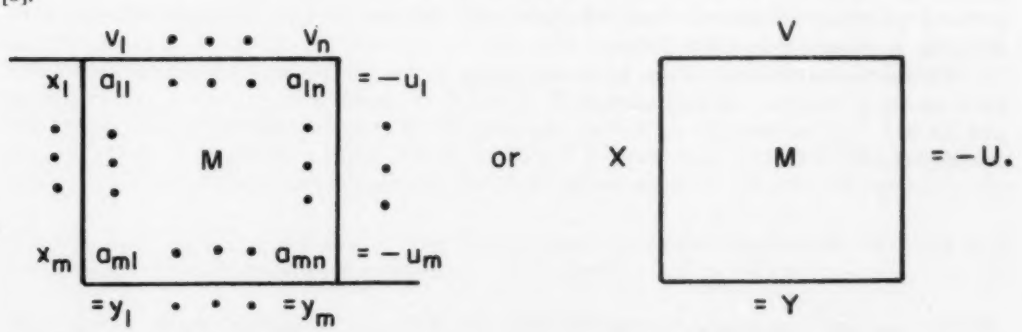
Key words: Combinatorial equivalence; linear inequalities; linear programs; pivot operations; skew-symmetry.

A finite class of skew matrices can be associated with a self-dual linear system. It is known [1, 2][1] that each such class contains one or more skew matrices with one type of "saddlepoint," a sign configuration consisting of a nonnegative row and corresponding nonpositive column. Using this saddlepoint theorem, we prove a similar result with the reverse sign pattern.

An open question is whether such a class of matrices contains a skew matrix that simultaneously exhibits both a nonnegative row and a nonpositive row. We can prove that this "double saddlepoint" property does hold for skew matrices of small order and for arbitrarily large finite degenerate cases, but the general problem remains unsolved.

1. P-Pivots and the First Saddlepoint Theorem

Dual systems of linear equations can be represented in the following tableau format of Tucker [5]:



AMS Subject Classification: 15A39, 15A21.

* This work was completed while the author was a National Academy of Sciences-National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. This paper is dedicated to Professor A. W. Tucker on the occasion of his retirement from Princeton University.

[1] Figures in brackets indicate the literature references at the end of this paper.

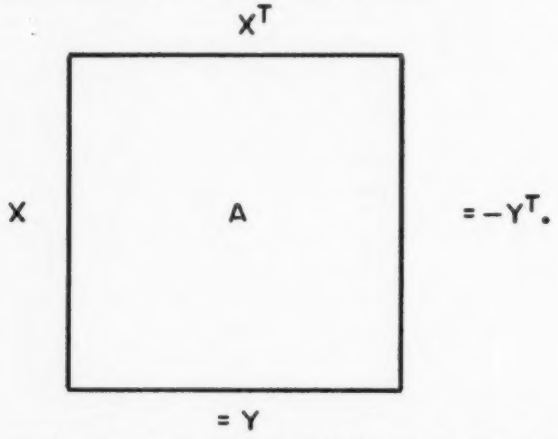
Here X, Y, U^T and V^T are row vectors. The tableau simultaneously represents the column system $XM=Y$ and the row system $MV=-U$. The systems are always *dual* in the sense of orthogonality, for

$$XU+YV=X(-MV)+(XM)V=0.$$

Since this paper is concerned with the sign patterns in matrices, we necessarily allow variables and coefficients to take values only in an ordered field. The symbols $\oplus$ and $\ominus$ respectively denote nonnegative and nonpositive elements.

DEFINITION 1: An $n\times n$ matrix A is *skew* when $A=-A^T$.

Throughout this paper A denotes a skew matrix of order n. Over an ordered field the diagonal entries of A must be zero. The tableau of A is written as follows:



By negative transposition it is clear that any *solution* (X, Y) to the column system $XA=Y$ is also a row solution (Y, X) to $AX^T=-Y^T$ and conversely. Hence each of the linear systems is *self-dual* because X and Y must be orthogonal.

DEFINITION 2: A *principal pair pivot* or *P-pivot* in (the tableau of) a self-dual linear system is a simultaneous exchange of variables x_i with y_i and x_j with y_j, provided $a_{ij}=-a_{ji}\neq 0$, while leaving the remaining variables fixed.

The P-pivot involves solving the ith column (row) equation for x_j and the jth equation for x_i, substituting the resulting expressions into the remaining equations. At the same time there occurs a (principal) permutation that exchanges row i with row j and column i with column j while fixing the remaining rows and columns.

We say that we *P-pivot on* a_{ij}, a_{ji} *in the matrix* A. For $a_{ij}\neq 0$, this operation yields another skew matrix $\bar{A}$ with the self-dual solution $(\bar{X}, \bar{Y})$ to $\bar{X}\bar{A}=\bar{Y}$ such that $\bar{x}_i=y_i$, $\bar{x}_j=y_j$, $\bar{y}_i=x_i$, $\bar{y}_j=x_j$ and for all $k\neq i, j$ we have $\bar{x}_k=x_k$, $\bar{y}_k=y_k$. The new skew matrix $\bar{A}$ is given by the following entries (recall that $i\neq j$):

$$\bar{a}_{ij}=-a_{ij}^{-1}.$$

$$\bar{a}_{ih}=-a_{ij}^{-1}a_{jh} \qquad \text{for } h\neq i, j.$$

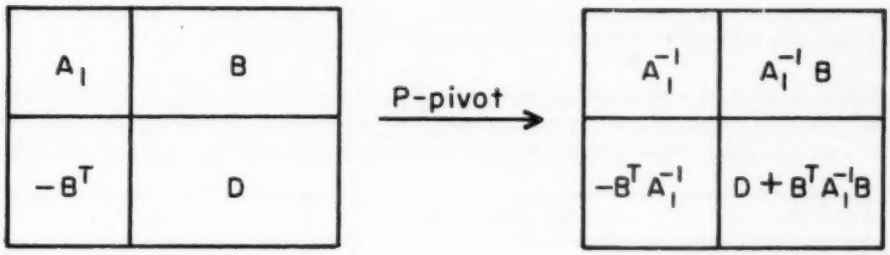
$$\bar{a}_{kj}=a_{ki}a_{ij}^{-1} \qquad \text{for } k\neq i, j.$$

$$\bar{a}_{kh}=a_{kh}-a_{ij}^{-1}(a_{ih}a_{kj}-a_{ki}a_{jh}) \qquad \text{for } k\neq i, j \text{ and } h\neq i, j.$$

$$\bar{a}_{tt}=0 \qquad \text{for all } t.$$

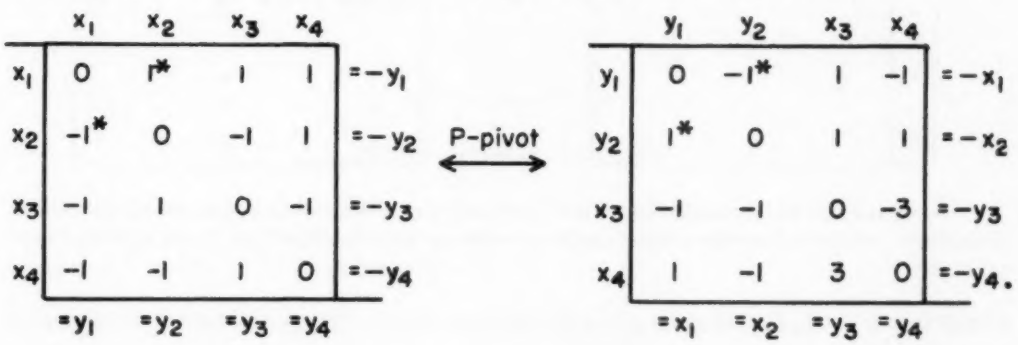
Checking the above formulae requires only straightforward algebraic manipulations.

Schematically, a P-pivot or sequence of P-pivots corresponds to inversion of a nonsingular principal submatrix of A and the appropriate extensions to the rest of A:



A principal permutation places the nonsingular submatrix A_1 in the upper corner.

EXAMPLE: In this 4×4 self-dual tableau we show a P-pivot on the starred entries:



Sample solution: $(X, Y) = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (1, 0, -1, 0, 1, 0, 1, 2)$.
$(\bar{X}, \bar{Y}) = (y_1, y_2, x_3, x_4, x_1, x_2, y_3, y_4) = (\bar{x}_1, \ldots, \bar{x}_4, \bar{y}_1, \ldots, \bar{y}_4)$.

P-pivots preserve both skewsymmetry and solution sets. The new tableau has the same solution set as the original one in the sense that values assigned to the variables in one tableau will hold for the same variables in the other one even though their positions may have changed.

From a given skew matrix of order n, only a finite number of skew matrices can be obtained by a sequence of P-pivots [1, Theorem 2.2; 2, Theorem 2]. Such matrices are in fact a subset of Tucker's "combinatorial equivalence class" [4] for the given matrix, and they are called *P-pivot transforms*.

DEFINITION 3: A nontrivial solution (X, Y) of $XA = Y$ is *elementary* if the only other nontrivial solutions having the same zero components as (X, Y) are of the form (kX, kY) for scalars $k \neq 0$.

DEFINITION 4: *A zero saddlepoint of type (1)* in a matrix is a sign configuration consisting of a nonnegative row and a nonpositive column.

THEOREM 1: (First Saddlepoint Theorem) *Given a skew matrix* A *of order* n *and an index* h, $1 \leq h \leq n$, *there exists a nonnegative elementary solution of* $XA = Y$ *such that* $x_h + y_h > 0$, *and there exists a skew matrix* $\bar{A}$ *which is a P-pivot transform of* A *and such that* $\bar{a}_{hj} \geq 0$ *for all* j, $1 \leq j \leq n$.

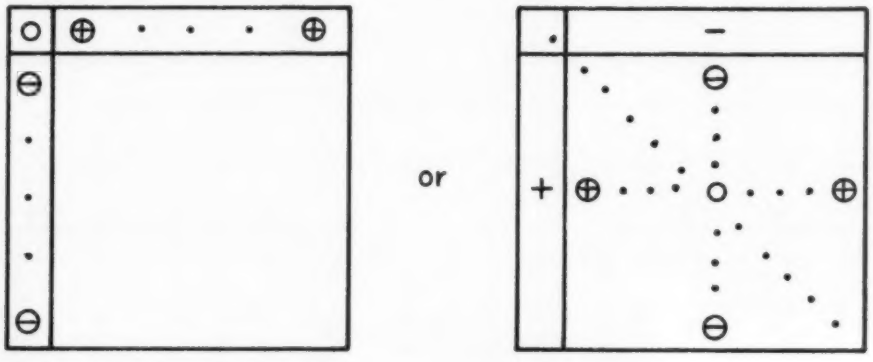
Theorem 1 says the existence of a type (1) saddlepoint occurs simultaneously with a non-

negative elementary solution and gives the theorem its name. Since the proof of Theorem 1 is presented elsewhere in full detail [1, Theorem 4.2; 2, Theorem 3], we omit it here.

2. The Second Saddlepoint Theorem

A result related to Theorem 1 is the following [1, Theorem 3.1]:

THEOREM 2: *Let* A *be a skew matrix of order* n. *By a finite sequence of* P*-pivots outside the first row and column of* A *we can obtain a skew matrix* $\bar{A}$ *with either a nonnegative first row or a nonnegative row whose first column entry is positive. Schematically, we obtain one of the following:*

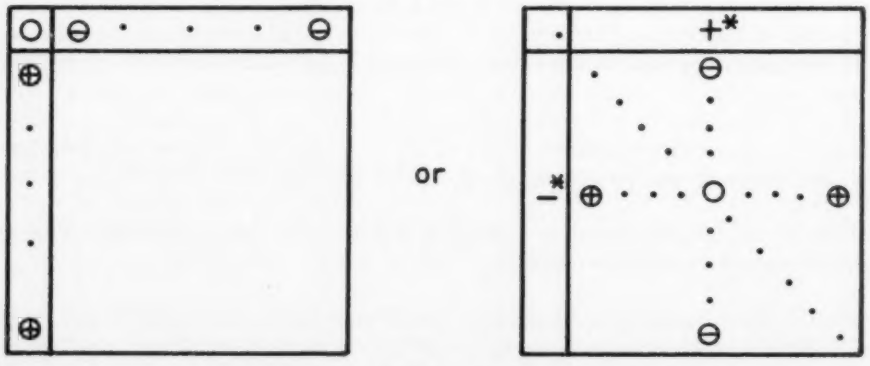


Theorem 2 can be generalized to *P*-pivots outside any given row and the corresponding column. Hence the first row and column can be considered the "distinguished" ones without loss of generality.

DEFINITION 5: *A zero saddlepoint of type (2)* in a matrix is a sign configuration consisting of a nonpositive row and a nonnegative column.

THEOREM 3: *Let* A *be a skew matrix of order* n *and let* h *be an index,* $1 \leq h \leq n$. *By a finite sequence of* P*-pivots outside the* h*th row and column of* A *we can obtain a skew matrix* $\bar{A}$ *with either a nonpositive* h*th row or a row in which the* h*th column entry is negative and every other entry is nonnegative.*

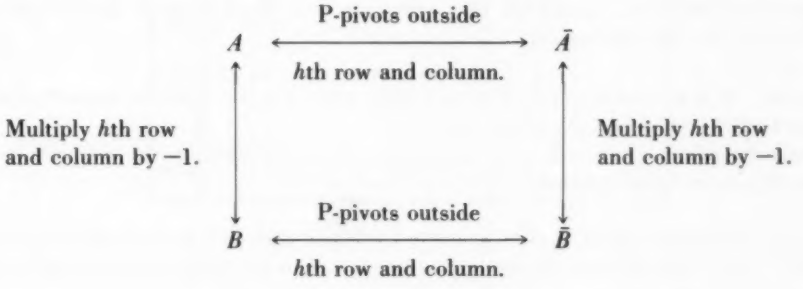
Schematically, we obtain one of the following for $h = 1$:



A *P*-pivot on the starred entries in the second alternative yields a saddlepoint of type (2) in the distinguished row and column.

PROOF: Given the skew matrix A, form the skew matrix B by multiplying the distinguished row and corresponding column of A by -1, leaving the rest of A unchanged. A P-pivot on a_{ij}, a_{ji} outside the distinguished row and column of A corresponds to a P-pivot on b_{ij}, b_{ji} in B. From the formulae following Definition 2 we can easily obtain the entries of the P-pivot transform $\bar{B}$ in terms of the corresponding entries of $\bar{A}$. When h is the index for the distinguished row and column, $a_{ht} = -b_{ht}$ and $a_{th} = -b_{th}$ for all t, $1 \leq t \leq n$. Otherwise $a_{kt} = b_{kt}$. Hence $\bar{B} = \bar{A}$ except for the distinguished row and column, where $\bar{b}_{ht} = -\bar{a}_{ht}$ and $\bar{b}_{th} = -\bar{a}_{th}$ for all t.

Thus the same sequence of P-pivots outside a distinguished row and column in A that produces a saddlepoint of type (1) will produce a saddlepoint of type (2) in the corresponding matrix B and vice versa. Equivalently, we may observe that the following diagram is "commutative":



(Same pivot choices as above.)

Combining these observations with the general form of Theorem 2 proves the theorem.

To obtain an analog of Theorem 1 we need to obtain a type of solution whose existence is equivalent to having a type (2) saddlepoint.

DEFINITION 6: A *onenegative solution* in a self-dual tableau is a solution (X, Y) to $XA = Y$ having precisely one negative component.

DEFINITION 7: A *complementary solution* in a self-dual tableau is a solution (X, Y) to $XA = Y$ such that $x_i y_i = 0$ for each i, $1 \leq i \leq n$.

REMARK: A onenegative complementary solution (X, Y) to $XA = Y$ is of the form:

$$x_j + y_j < 0 \qquad \text{for one index } j.$$

$$x_i + y_i \geq 0 \qquad \text{for } i \neq j,\ 1 \leq i \leq n.$$

$$x_j y_i = 0 \qquad \text{for each } i,\ 1 \leq i \leq n.$$

Multiplying the hth row and column entries in the matrix of a self-dual tableau by -1 changes the sign of the left-hand sides of the hth row and column equations and of the coefficients of x_h in the remaining equations. Hence the original system may be preserved by simultaneously changing the signs of y_h and x_h. Thus any nonnegative complementary solution associated with A would become onenegative when associated with the corresponding B; and conversely a onenegative complementary solution with $x_h + y_h < 0$ would become nonnegative. With the additional observation that any nonnegative solution of XA = Y must be complementary, we have proved the following result.[2]

[2] I am indebted to Dr. A. J. Goldman for suggesting this method of proof.

THEOREM 4: (Second Saddlepoint Theorem) *Given a skew matrix* A *of order* n *and an index* h, $1 \leq h \leq n$, *there exists a onenegative complementary solution of* $XA = Y$ *such that* $x_h + y_h < 0$, *and there exists a skew matrix* $\overline{A}$ *which is a* P*-pivot transform of* A *and such that* $\overline{a}_{hj} \leq 0$ *for all* $j, 1 \leq j \leq n$.

Theorem 4 can also be proved independently in a fashion similar to a proof of Theorem 1 [2]. The First and Second Saddlepoint Theorems are logically equivalent.

3. A Double Saddlepoint Problem

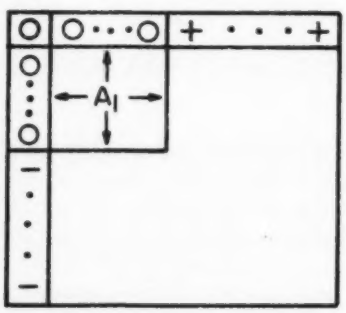
One question that naturally arises is whether any skew A has a P-pivot transform $\overline{A}$ having saddlepoints of both types (1) and (2). One may even ask whether *every* skew A has a P-pivot transform with a "double saddlepoint."

CONJECTURE: If A is a skew matrix of order n, then there exists a P-pivot transform $\overline{A}$ of A having both a nonnegative row and a nonpositive row.

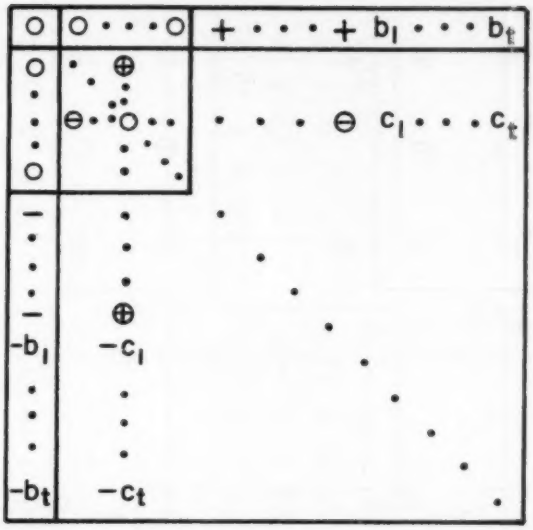
A single row of zeros will satisfy the conjecture. It is proved below for arbitrarily large degenerate cases (Theorem 5) and in general for $n \leq 5$ (Theorem 6).

THEOREM 5: *If a skew matrix* A *of order* n *can be transformed by a finite number of* P*-pivots into a skew matrix with a saddlepoint of type (1) having more than one zero entry in the nonnegative row, then* A *satisfies the Conjecture.*

PROOF: By Theorem 1, we can always obtain a P-pivot transform with a nonnegative row from a given skew matrix A of order n. If the nonnegative row is all zero, there is nothing to prove. Hence assume the nonnegative row has at least two zero entries and at least one positive entry. By a principal permutation we obtain:



where A_1 is a nonempty $k \times k$ principal skew submatrix of A, $1 \leq k < n$. (The inverse permutation can always be applied later.) By Theorem 4 we can reach a saddlepoint of type (2) in A_1 by a finite number of P-pivots in A_1 (i.e., P-pivots in A with pivot entries chosen only from A_1). Noting that P-pivots in A_1 do not affect the first row or column of A, we obtain:



where b_i, $c_i > 0$ for $1 \leq i \leq t$. If the collection of b's and c's is empty, there is nothing left to do. Otherwise P-pivot on b_j, $-b_j$ where $c_j/b_j = \max_i (c_i/b_i)$ to reach the desired form.

To prove the Conjecture for skew matrices of order $n \leq 5$, two lemmas will be helpful.

LEMMA 1: *If a skew matrix* A *of order* n > 1 *can be transformed by a finite number of* P*-pivots into a skew matrix with a nontrivial nonnegative row, and if some positive entry* a_{ij} *in that row is distinguished by being the only positive entry in its column, then* A *satisfies the Conjecture.*

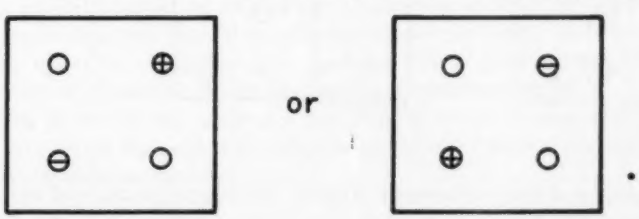
PROOF: P-pivot in the nonnegative row and nonpositive column on the distinguished positive and negative entries a_{ij}, a_{ji} to reach the desired form.

LEMMA 2: *Suppose a skew matrix* A *of order* n > 1 *can be transformed by a finite number of* P*-pivots into a skew matrix with a nontrivial nonnegative* ith *row containing* $a_{ij} > 0$. *If there exists another entry* a_{kj} *in the* jth *column which, except for* a_{ij}, *is the sole positive entry in the* kth *row and* jth *column, then* A *satisfies the Conjecture.*

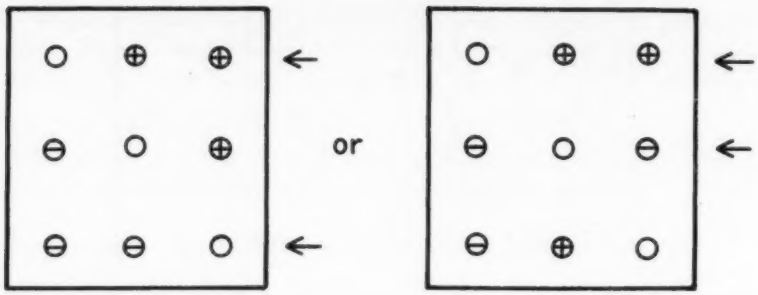
PROOF: P-pivot on a_{ij}, a_{ji} to reach the desired form.

THEOREM 6: *If* A *is a skew matrix of order* n ≤ 5, *then* A *has a* P*-pivot transform* $\bar{A}$ *with both a nonnegative row and a nonpositive row.*

PROOF: For $n = 1$ and $n = 2$ the theorem is clearly true since the only possibilities are [0] and

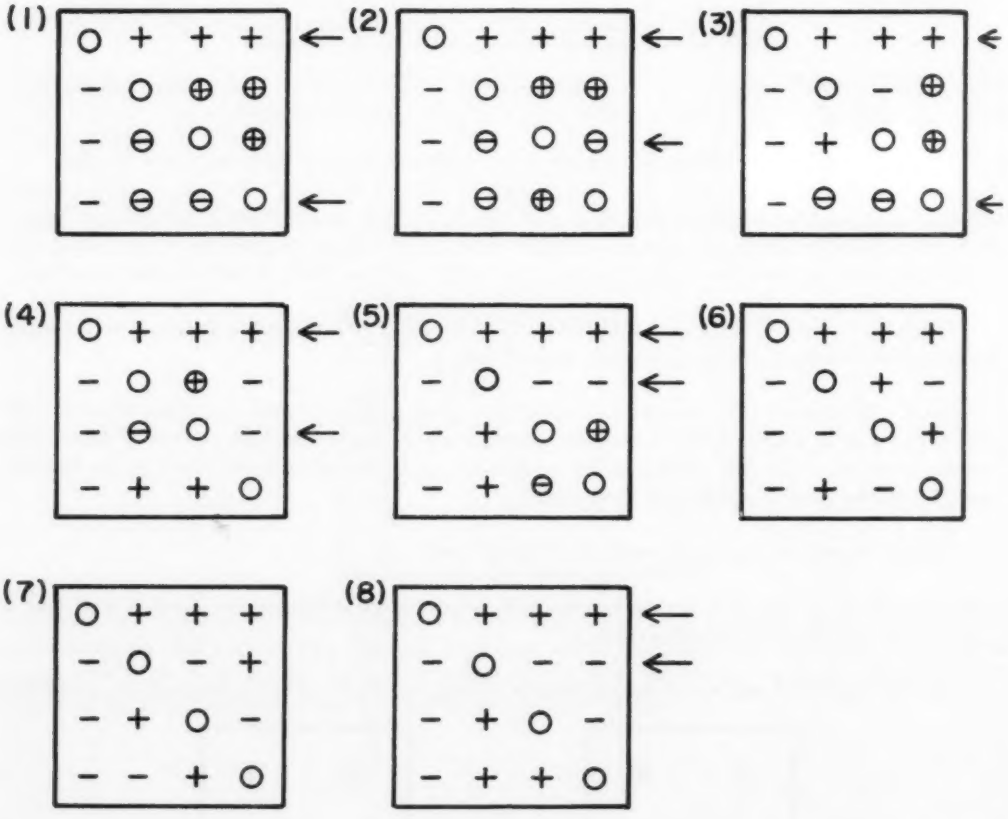


By Theorem 1, we need consider only those cases for $n = 3$ where we already have a nonnegative row. The two possibilities are:



with arrows indicating the desired rows.

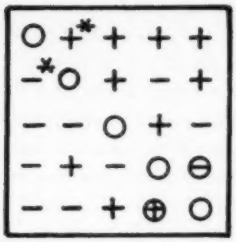
For $n = 4$ we apply Theorems 1 and 5, leaving only the eight cases having a nonnegative row with only one zero entry:



The first five cases and case (8) already possess the desired form, and the remaining cases (6) and (7) satisfy Lemma 2.

For $n = 5$, application of Theorems 1 and 5 again limit the cases to be considered. Lemma 1 eliminates other possibilities, narrowing the choices to the following nine sign patterns:

(1)



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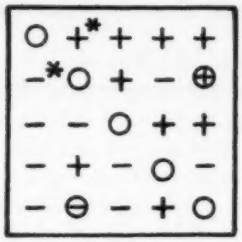

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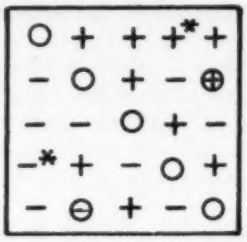

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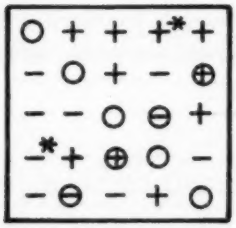

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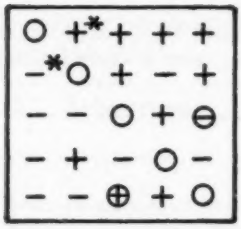
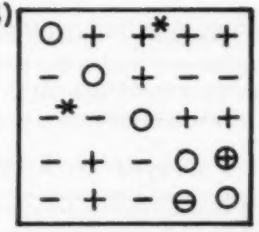
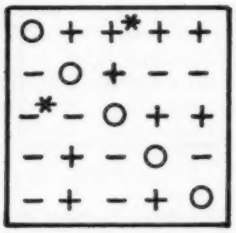


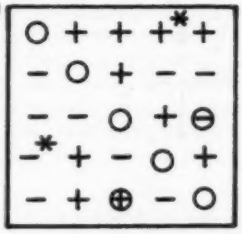
(6)



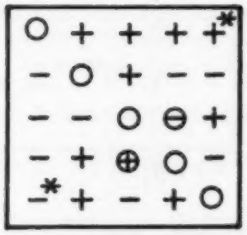
(7)



(8)



(9)



By Lemma 2, P-pivots on the starred entries will yield a double saddlepoint in each case.

4. Applications

The First and Second Saddlepoint Theorems have many applications in the theory of dual linear systems and related topics. Both theorems are logically equivalent to Tucker's Skew-Symmetric Matrix Theorem [1, Theorem 5.1; 2, Corollary 1]. From this result the main theorems of linear programming can be elegantly derived [1, Chapter 8]. Other applications include the theory of matrix games [1, Chapter 6] and classical theorems of Gordan, Stiemke, Farkas and von Neumann on linear inequalities [1, Chapter 9].

In view of these applications of the First and Second Saddlepoint Theorems, their generalization to the Double Saddlepoint Conjecture can be expected to have interesting consequences. One consequence of the proved degenerate case of the Conjecture is presented below. Although the Conjecture is still unproved in the general nondegenerate case, a recent undergraduate project at Princeton under the direction of Tucker has examined the P-pivot classes of 100 randomly-generated skew matrices of orders 6 to 10 and has yielded no counterexample. [3]

Theorem 5, which shows the existence of a double saddlepoint for skew matrices if one of the single saddlepoints has an off-diagonal zero, will now be shown to have a surprising consequence for simple pivots in rectangular matrices.

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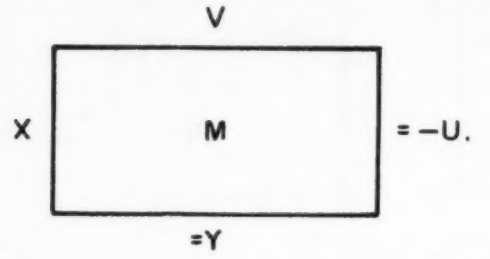
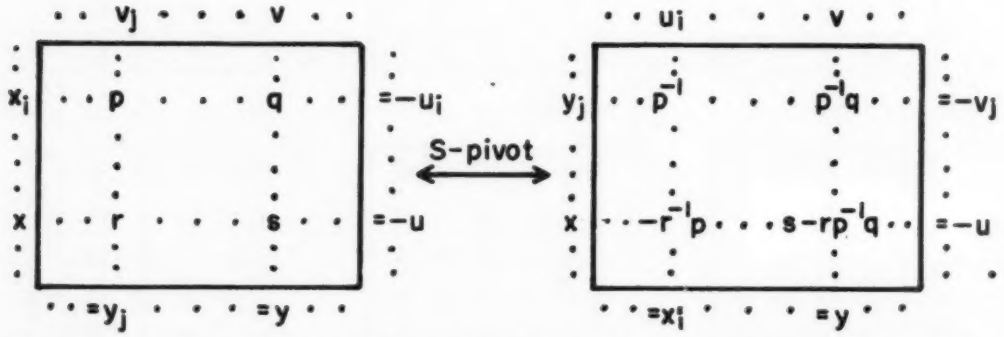
[3] Personal communication from Professor Tucker.

DEFINITION 8: *Let* M *be an* m × n *matrix over an ordered field with the tableau*



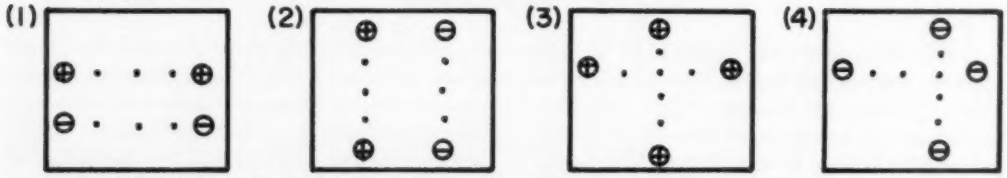
A *simple pivot* or *S-pivot* in (the tableau of) the dual linear systems $XM = Y$ and $MV = -U$ is a simultaneous exchange of variables x_i with y_j and u_i with v_j, provided $m_{ij} \neq 0$, while leaving the remaining variables fixed.

Schematically, for $p = m_{ij} \neq 0$:



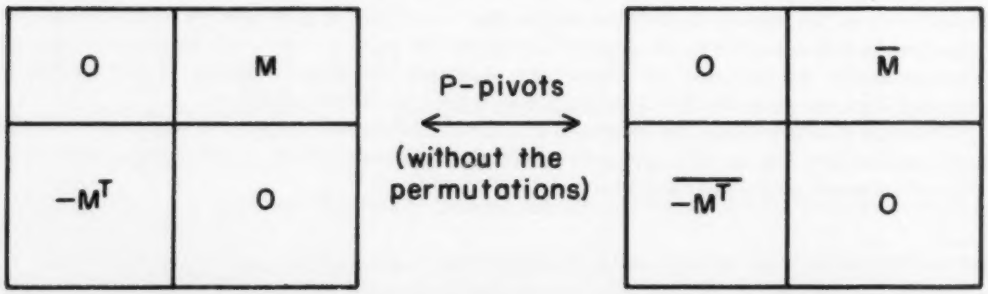
THEOREM 7: *Given any rectangular matrix* M *over an ordered field, by a finite number of simple pivots in* M *we can reach a matrix* $\overline{\mathrm{M}}$ *having either a nonnegative row and a nonpositive row, or a nonnegative column and a nonpositive column, or a row and column of the same sign.*

Schematically, Theorem 7 says we can reach one of the following:



Note that the presence of a trivial row or column immediately gives us (1) or (2).

PROOF: Given M, form an expanded skew tableau with $-M^T$ and blocks of zeros:



If M has only one row or column, it is already in form (1), (2), (3) or (4), so assume M has at least two rows and two columns.

By Theorem 1, a finite sequence of P-pivots in the expanded tableau yields a type (1) saddlepoint. If for each P-pivot we perform the inverse permutation, then $\bar{M}$ will remain in the upper right-hand corner, as though the P-pivot did not include a permutation. Such modified P-pivots correspond to pairs of S-pivots on m_{ij} in M and $-m_{ji}$ in $-M^T$. Moreover, pivots in M or in $-M^T$ preserve the blocks of zeros. Hence by Theorem 5 we can repeatedly P-pivot in the expanded tableau (S-pivot in M, $-M^T$) to reach a double saddlepoint.

If both rows of opposite sign appear in $\bar{M}$, then we have form (1) in $\bar{M}$. If both rows appear in $\overline{-M^T}$, then we have reached (2) in $\bar{M}$. One row in $\bar{M}$ and the other in $\overline{-M^T}$ yields (3) or (4) in $\bar{M}$, proving the theorem.

From its form it appears that Theorem 7 may lead to further results in the theories of positive semidefinite quadratic programs and of bimatrix games.

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Maximizing the Number of Spanning Trees in a Graph With n Nodes and m Edges*

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(September 4, 1974)

The problem considered is that of determining, among all graphs on n nodes and m edges, those having the maximum number of spanning trees. The possible candidate graphs can be obtained by deleting some number k of edges from a complete n-node graph. For $k \leq n/2$, it is shown that the maximum occurs when the k edges are mutually nonadjacent.

Key words: Combinatorial analysis; enumeration; graphs; maximization; spanning trees; trees

Consider the class of undirected graphs having n nodes and m edges. The problem to be addressed here is that of finding specific configurations of m edges on the given n nodes so that the resulting graph will contain the largest number of spanning trees. In particular, an explicit solution to this problem will be exhibited for graphs which have "enough" edges.

To be specific, let the set E of k edges be deleted from the complete graph K_n on n nodes: K_n has an edge between every pair of distinct nodes and thus contains $n(n-1)/2$ edges. For the case when $k \leq n/2$, we will demonstrate that the number of spanning trees $T(n, E)$ in the resulting graph is maximized by choosing the k deleted edges to be mutually nonadjacent. The (apparently more complicated) cases with $k > n/2$ await resolution.

Let P_k denote a set of k nonadjacent ("parallel") edges in K_n, where $k \leq n/2$. We will show that $|E| = k$ implies $T(n, E) \leq T(n, P_k)$. First the case when the edge set E is disconnected will be disposed of. This will be done in the context of the inductive hypothesis that if $i < k$ and $|S| = i$, then $T(n, S) \leq T(n, P_i)$ — whether or not S is connected. Suppose $|E| = k$ and that E can be decomposed into connected edge sets $C_1, C_2, \ldots, C_p$ with $p \geq 2$. Certainly, when $k = 2$ the only disconnected set E possible consists of two nonadjacent edges, so that the inductive hypothesis holds. More generally, if $|E| = k$ then $|C_1| < k$ and $n \geq 2k > 2|C_1|$, whence $0 \leq T(n, C_1) \leq T(n, P_\alpha)$, $\alpha = |C_1|$, by the inductive hypothesis. Similarly, we have $0 \leq T(n, \bar{C}_1) \leq T(n, P_\beta)$, $\beta = |\bar{C}_1|$, where $\bar{C}_1 = C_2 \cup \ldots \cup C_p$ in the decomposition $E = C_1 \cup C_2 \cup \ldots \cup C_p$. From [2, p. 106] it is known that

$$T(n, P_i) = n^{n-2}\left(1 - \frac{2}{n}\right)^i,$$

whereupon

$$T(n, C_1)T(n, \bar{C}_1) \leq n^{n-2}n^{n-2}\left(1 - \frac{2}{n}\right)^k.$$

Also, the following relation [2, p. 106] obtains for disjoint edge sets:

$$T(n, C_1)T(n, \bar{C}_1) = n^{n-2}T(n, C_1 \cup \bar{C}_1),$$

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*This work was done while the author was a National Academy of Sciences-National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234.

from which it is immediate that

$$T(n, E) \leq n^{n-2}\left(1-\frac{2}{n}\right)^{k} = T(n, P_k).$$

Subsequently, then, it will be assumed that E is a connected set of k edges on r nodes. It is only necessary to consider the case when $r > 2$ and show that $T(n, E) \leq T(n, P_k)$; if $r = 2$ then $k = 1$ and this relation is immediately satisfied with equality. The following result is most helpful in establishing the relation for $r > 2$; it involves a symmetric function ϕ for which the number of arguments, as well as their values, will be considered variable.

LEMMA: *Suppose that* $\phi(\mathbf{d}) = \phi(d_1, \ldots, d_r) \equiv \prod_{i=1}^{r}\left(1-\frac{d_i}{n}\right)$ *with* $\sum_{i=1}^{r} d_i = 2k$, $n \geq 2k$, $k+1 \geq r > 2$ *and all* d_i *positive integers. Then* $\phi(\mathbf{d})$ *is maximized over all* $\mathbf{d} \neq (1, 1, 2, 2, \ldots, 2)$ *by* $\mathbf{d}_0 = (2, 2, \ldots, 2)$.

PROOF: Given a sequence $\mathbf{d}$, execute the following procedure.

PROCEDURE REDUCE.

1. Let d_i = a smallest element of $\mathbf{d}$,
   d_j = a largest element of $\mathbf{d}$.
2. If $d_j - d_i > 1$ then $d_i := d_i + 1$, $d_j := d_j - 1$ and go to Step 1. Otherwise, terminate.

Note that this procedure is always finite and does not change the values for r and $\sum_{i=1}^{r} d_i$. Moreover, if the procedure yields as output the vector $\mathbf{d}^*$ for the input sequence $\mathbf{d}$, then $\phi(\mathbf{d}) \leq \phi(\mathbf{d}^*)$. In fact

$$\left(1-\frac{d_i}{n}\right)\left(1-\frac{d_j}{n}\right) \leq \left(1-\frac{d_i+1}{n}\right)\left(1-\frac{d_j-1}{n}\right)$$

$$\Leftrightarrow d_i d_j \leq (d_i+1)(d_j-1)$$
$$\Leftrightarrow 1 \leq d_j - d_i.$$

Thus at each step of REDUCE the value of ϕ cannot decrease, and so $\phi(\mathbf{d}) \leq \phi(\mathbf{d}^*)$. Upon termination of this procedure, one of the following final forms for $\mathbf{d}^*$ will be obtained (to within permutation):

(i) $(1, 1, \ldots, 1)$
(ii) $(s, s, \ldots, s), s > 1$
(iii) $(s, \ldots, s, s+1, \ldots, s+1), s > 1$
(iv) $(1, \ldots, 1, 2, \ldots, 2)$.

In Case (i), $1 + 1 + \ldots + 1 = r = 2k$. However, $r \leq k + 1$ implies $2k \leq k + 1$ so that $k = 1$. This case cannot then occur since it is assumed that $k + 1 > 2$.

In Case (ii), consider the situation when s is even. The following relation will allow a further simplification of $\mathbf{d}^*$. Namely, if $d_i \geq 2$ then

$$\left(1-\frac{d_i}{n}\right) \leq \left(1-\frac{2}{n}\right)\left(1-\frac{d_i-2}{n}\right). \tag{1}$$

(The above inequality is valid since $2(d_i - 2) \geq 0$.) Therefore, by replacing each s (if $s > 2$) with $(2, s-2)$, the value of ϕ cannot decrease and so eventually $\phi(s, \ldots, s) \leq \phi(2, \ldots, 2)$. If s is odd

$(s \geq 3)$, then the simplification $s := (2, s-2)$ eventually results in $\mathbf{d}_1 = (2, \ldots, 2, 3, \ldots, 3)$, where the number t of such 3's is necessarily even since $\Sigma d_i = 2k$. For each pair (3, 3) the following inequality holds:

$$\left(1-\frac{3}{n}\right)\left(1-\frac{3}{n}\right) \leq \left(1-\frac{2}{n}\right)^3, \tag{2}$$

because $n \geq 2k > 2$ implies $9n \leq 12n - 8$. Together with (1) the above inequality shows that $\phi(\mathbf{d}^*) \leq \phi(\mathbf{d}_1) \leq \phi(2, 2, \ldots, 2)$. In any event, then, $\phi(\mathbf{d}) \leq \phi(\mathbf{d}^*) \leq \phi(\mathbf{d}_0)$.

Suppose that s is odd in Case (iii), so that there are an even number of these s values in $(s, \ldots, s, s+1, \ldots, s+1)$. As in Case (ii) the simplifications implied by (1) and (2) yield $\phi(\mathbf{d}^*) \leq \phi(2, 2, \ldots, 2)$. A similar argument applies when s is even.

In Case (iv), suppose that the number of 1's is t, so the number of 2's is $r - t$. Now,

$$2r - 2 \leq 2k = \sum_{i=1}^{r} d_i = t + 2(r-t) = 2r - t,$$

whence $t \leq 2$. Since t must be even, either $t = 0$ or $t = 2$. In the former situation we have the vector $(2, 2, \ldots, 2)$ which is already subsumed under Case (ii); in the latter, we have $\mathbf{d}^* = (1, 1, 2, 2, \ldots, 2)$. Note that unless $\mathbf{d} = \mathbf{d}^*$ this possibility must have arisen from applying Step 2 of REDUCE to the vector $\mathbf{d}_2 = (1, 1, 1, 2, 2, \ldots, 2, 3)$, which in turn arose from zero or more applications of Step 2 starting with the original $\mathbf{d}$. Thus $\phi(\mathbf{d}) \leq \phi(\mathbf{d}_2)$.

However,

$$\phi(\mathbf{d}_2) = \left(1-\frac{1}{n}\right)^3\left(1-\frac{3}{n}\right)\left(1-\frac{2}{n}\right)^{k-3}$$

$$\phi(\mathbf{d}_0) = \left(1-\frac{2}{n}\right)^k$$

so that

$$\phi(\mathbf{d}_2) \leq \phi(\mathbf{d}_0) \Leftrightarrow \left(1-\frac{1}{n}\right)^3\left(1-\frac{3}{n}\right) \leq \left(1-\frac{2}{n}\right)^3$$

$$\Leftrightarrow 8n + 3 \leq 10n.$$

Since $n \geq 2k > 2$ this latter relation certainly holds and so $\phi(\mathbf{d}_2) \leq \phi(\mathbf{d}_0)$. It therefore follows that if $\mathbf{d} \neq (1, 1, 2, 2, \ldots, 2)$ then $\phi(\mathbf{d}) \leq \phi(\mathbf{d}_0)$ in Case (iv).

What is concluded from these four cases is that $\phi(\mathbf{d}) \leq \phi(\mathbf{d}_0)$ for all $\mathbf{d} \neq (1, 1, 2, 2, \ldots, 2)$, which is precisely the result to be established in the Lemma. It should be noted that the degrees d_i of a connected graph with $r > 2$ nodes and k edges satisfy the hypotheses of the Lemma if $n \geq 2k$.

Now we are in a position to prove the main result. Let $\{1, 2, \ldots, n\}$ be the nodes of K_n, and let $\{1, 2, \ldots, r\}$ be the nodes of a connected set E of k edges, with $r > 2$ and $n \geq 2k$. Also, let d_i denote the degree in E of node i. There are two cases to consider.

(i) E forms a chain. Then from [2, p. 108]

$$T(n, E) = n^{n-2} \sum_{j=0}^{k} \binom{2k-j+1}{j} \left(-\frac{1}{n}\right)^j$$

$$\leq n^{n-2}\left(1-\frac{2}{n}\right)^k,$$

using a combinatorial inequality proved in [4]. Thus $T(n, E) \leq T(n, P_k)$ and so the desired relation is obtained.

(ii) E does not form a chain. Then the degree sequence $\mathbf{d} = (d_1, \ldots, d_r) \neq (1, 1, 2, 2, \ldots, 2)$. Since E is a connected set of edges, the hypotheses of the above lemma are satisfied and so it is concluded that $\phi(\mathbf{d}) \leq \phi(\mathbf{d}_0) = \left(1 - \frac{2}{n}\right)^k$. Moreover, an expression for $T(n, E)$ is given [6] by $T(n, E) = n^{n-2} \det A$, where $A = (a_{ij})$ is the $r \times r$ matrix defined by

$$a_{ij} = \begin{cases} 1 - \dfrac{d_i}{n} & \text{if } i = j \\ \dfrac{1}{n} & \text{if } i \neq j,\ (i, j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Now since E has k edges, $d_i \leq k \leq n/2$ or $1 - \frac{d_i}{n} \geq \frac{1}{2}$; thus $a_{ii} \geq \sum_{j \neq i} |a_{ij}|$ and accordingly matrix A is positive semidefinite. Also, for some i, $\frac{d_i}{n} < \frac{1}{2}$ since otherwise

$$\frac{nr}{2} = \sum_{i=1}^{r} d_i = 2k \leq n$$

or $r \leq 2$, a case which has been previously excluded. Inasmuch as A is symmetric and for some i, $a_{ii} > \frac{1}{2} > \sum_{j \neq i} |a_{ij}|$, it follows [5] that A is nonsingular and hence positive definite. Furthermore, using the Hadamard bound on det A [1, p. 129], we obtain

$$T(n, E) = n^{n-2} \det A \leq n^{n-2} \phi(\mathbf{d}) \leq n^{n-2} \left(1 - \frac{2}{n}\right)^k = T(n, P_k).$$

Thus in either case $T(n, E) \leq T(n, P_k)$ and so the number of spanning trees is maximized by choosing the k edges to be mutually nonadjacent. Of course, in general there are numerous ways of choosing k parallel edges to delete from K_n, all of which result in the same (maximum) number of spanning trees. The restriction that $n \geq 2k$ is certainly a natural one, since it ensures the possibility of being able to select k parallel edges from the n-node complete graph. Finding the maximum (or minimum) number of spanning trees when $k > n/2$ appears to be an open problem [3].

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The Field of Values and Spectra of Positive Definite Multiples*

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(September 11, 1974)

Suppose $0 \neq \lambda \epsilon C$ and $A \epsilon M_n(C)$. We show constructively that λ is an eigenvalue of HA for some $H^* = H > 0$ if and only if $\lambda = x^*Ax$ for some $x \epsilon C^n$. The characterization of H-stable matrices is then an easy corollary.

Key words: Eigenvalues; field of values; H-stable; positive definite matrix; spectrum.

Let $\sigma(A)$ denote the set of all eigenvalues of $A \epsilon M_n(C)$, the n by n complex matrices, and let

$$F(A) \equiv \{x^*Ax : x \epsilon C^n, x^*x = 1\},$$

the "field of values" of A.

In [1, 2, 4][1] the H-stable matrices were characterized. (A matrix $A \epsilon M_n(C)$ is called H-stable if $\lambda \epsilon \sigma(HA)$ implies $\mathrm{Re}(\lambda) > 0$ for all $H^* = H > 0$.) The simplest version of the characterization is that A is H-stable if and only if A is nonsingular and $\lambda \epsilon F(A)$ implies $\mathrm{Re}(\lambda) > 0$ or $\lambda = 0$. In this note we give a simple characterization of those $\lambda \epsilon \sigma(HA)$ for some $H^* = H > 0$ and the theorem on H-stability, for example, is an easy corollary. The characterization is constructive in that a specific class of positive definite matrices H for which $\lambda \epsilon \sigma(HA)$ is produced. Let $\Sigma \equiv \{H \epsilon M_n(C) : H^* = H > 0\}$.

We first make two observations:

(1) $0 \epsilon \sigma(HA)$ for some $H \epsilon \Sigma$ if and only if $0 \epsilon \sigma(KA)$ for all $K \epsilon \Sigma$ if and only if $0 \epsilon \sigma(A)$;

(2) if $H = BB^*$, B nonsingular, then $\sigma(HA) = \sigma(B^*AB)$.

THEOREM 1: *Suppose* $0 \neq \lambda \epsilon C$. *Then the following are equivalent for* $A \epsilon M_n(C)$:

(i) $\lambda = x^*Ax$, $x \epsilon C^n$;
(ii) $\lambda \epsilon \sigma(HA)$ *for some* $H \epsilon \Sigma$; *and*
(iii) $\lambda \epsilon \sigma(B^*AB)$ *for some nonsingular* $B \epsilon M_n(C)$.

PROOF: Since $H \epsilon \Sigma$ if and only if $H = BB^*$ for some nonsingular $B \epsilon M_n(C)$, it is clear from observation (2) that (ii) and (iii) are equivalent. To show that (iii) implies (i) suppose $\lambda \epsilon \sigma(B^*AB)$ with associated eigenvector y of length 1. Then $\lambda = y^*(B^*AB)y = (By)^*A(By) = x^*Ax$ for $x = By$. Conversely suppose $x^*Ax = \lambda$. Then $x \neq 0$ since $\lambda \neq 0$ and we let B_1 be any nonsingular matrix

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[1] Figures in brackets indicate the literature references at the end of this paper.

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in $M_n(C)$ whose first column is x. Then the 1, 1 entry of $B_1^* AB_1$ is λ. Let v be the vector formed by the remaining n-1 entries of the first row of $B_1^* AB_1$ and, in turn, $-z = \frac{1}{\lambda} v$. Then

$$B_2 = \left[\begin{array}{c|c} 1 & z \\ \hline 0 & I \end{array}\right]$$

is nonsingular and $B_2^*(B_1^* AB_1)B_2$

$$= \left[\begin{array}{c|c} \lambda & 0 \\ \hline * & * \end{array}\right]$$

so that $\lambda \epsilon \sigma(B^*AB)$, where $B = B_1B_2$ is nonsingular which completes the proof.

Let R denote the real field. In the same manner as theorem 1 we may also prove:

THEOREM 1′: *Suppose* $0 \neq \lambda \epsilon R$. *Then the following are equivalent for* $A \epsilon M_n(R)$:

(i′) $\lambda = x^T Ax$, $x \epsilon R^n$.

(ii′) $\lambda \epsilon \sigma(HA)$ *for some* $H \epsilon \Sigma \cap M_n(R)$; *and*

(iii′) $\lambda \epsilon \sigma(B^T AB)$ *for some nonsingular* $B \epsilon M_n(R)$.

We may exploit the above construction to give an explicit $H \epsilon \Sigma$ such that $0 \neq \lambda \epsilon \sigma(HA)$ assuming that $A = (a_{ij})$ and $x = (x_1, \ldots, x_n)$ such that $x^*Ax = \lambda$ are given. We assume without loss of generality that $x_1 \neq 0$ and define

$$B_1 \equiv \left[\begin{array}{c|c} x_1 & 0 \\ \hline x_2 & \\ \vdots & I \\ x_n & \end{array}\right], \quad B_2 \equiv \left[\begin{array}{c|c} 1 & -\frac{1}{\lambda}\sum_i \bar{x}_i a_{i2}, \ldots, \frac{1}{\lambda}\sum_i \bar{x}_i a_{in} \\ \hline 0 & I \end{array}\right]$$

Then $B = B_1B_2$ is nonsingular and $H = B_1B_2B_2^*B_1^* \epsilon \Sigma$ with $\lambda \epsilon \sigma\ (HA)$.

Let $\mathcal{P}$ denote the open right half-plane and we have:

COROLLARY: $A \epsilon M_n(C)$ *is* H-*stable if and only if* A *is nonsingular and* $F(A) \subset \mathcal{P} \cup \{0\}$.

PROOF: Suppose A is H-stable. Then A is nonsingular and, since (ii) implies (i) in theorem 1, it follows that $F(A) \subset \mathcal{P} \cup \{0\}$. Suppose, alternatively, that $\lambda \epsilon F(A)$ and $\lambda \notin \mathcal{P} \cup \{0\}$. Then $Re(\lambda) \not> 0$ and $\lambda \epsilon \sigma(HA)$ for some $H \epsilon \Sigma$. Thus A is not H-stable which completes the proof.

We close by mentioning a result parallel to theorem 1 which is proved elsewhere [3]. Let Γ denote an arbitrary open unbounded angular sector of the complex plane comprising less than 180 degrees. We then have

THEOREM 2: *The matrix* $A \epsilon M_n(C)$ *may be written as* $A = HB$ *where* $H \epsilon \Sigma$ *and* $F(B) \subset \Gamma$ *if and only if* $\sigma(A) \subset \Gamma$.

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Automatic Computing Methods for Special Functions. Part II. The Exponential Integral $E_n(x)$

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Accurate, automatic, efficient methods for computing the exponential integral $E_n(x)$ are detailed and implemented in an American National Standard FORTRAN program. The driver program and test results are also included.

Key words: Computer programs; continued fraction; exponential integral; key values; recurrence relation.

1. Introduction

The exponential integral itself occurs in many physical problems and often many other integrals are expressible in terms of the exponential integral. In view of its importance and the difficulties accompanying the repeated application of its recurrence relation, we have chosen this integral as Part II. (For Part I, see J. of Research NBS, Vol. **74B**, July-September 1970, pp. 211–224.)

Accurate, efficient, automatic computing methods implemented in American National Standard FORTRAN, covering the entire range of machine legitimate arguments and/or functional values will be supplied. The number of terms in series, the number of convergents in an iterative process, etc., are all determined by the computer as a function of word length, arguments, the accuracy desired, etc. In cases of error returns, more realistic results will be returned. To further ensure correct limiting values of related functions, the proper analytic behavior of the function will always be retained. The driver program and test results are also included.

2. Mathematical Formulas

Relevant formulas are collected here for completeness and ease of reference. In keeping with the convention of the Handbook [1], [1] x here is a real variable.

A. Definition

$$E_n(x) = \int_1^\infty t^{-n} e^{-xt} dt \qquad (n = 0, 1, 2, \ldots; x > 0)$$

$$= \int_0^1 u^{n-2} e^{-x/u} du$$

$$= x^{n-1} \int_x^\infty t^{-n} e^{-t} dt.$$

AMS Subject classification: 33–04, 65D20.

[1] Figures in brackets indicate references on page 205.

B. Series Expansion

$$E_n(x) = \frac{(-x)^{n-1}}{(n-1)!}[-\ln x + \Psi(n)] - \sum_{\substack{m=0 \\ m\neq n-1}}^{\infty}{}' \frac{(-x)^m}{m!\,(m-n+1)} \qquad (x>0)$$

$$\Psi(1) = -\gamma,\ \Psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m} \qquad (n>1)$$

γ(Euler's constant) = .57721 56649. . . .

C. Continued Fraction

$$E_n(x) = e^{-x}\left(\frac{1}{x+}\,\frac{n}{1+}\,\frac{1}{x+}\,\frac{n+1}{1+}\,\frac{2}{x+}\,\cdots\right) \qquad (x>0).$$

D. Asymptotic Expansion

$$E_n(x) \sim \frac{e^{-x}}{x}\left[1 - \frac{n}{x} + \frac{n(n+1)}{x^2} - \frac{n(n+1)(n+2)}{x^3} + \cdots\right].$$

E. Special Values

$$E_n(0) = \frac{1}{n-1} \qquad (n>1)$$

$$E_0(x) = \frac{e^{-x}}{x}.$$

F. Recurrence Relation

$$E_{n+1}(x) = \frac{e^{-x}}{n} - \frac{x}{n}E_n(x) \qquad n = 1, 2, 3, \ldots.$$

G. Differentiation Formula

$$\frac{d^k}{dx^k}E_n(x) = (-1)^k E_{n-k}(x) \qquad n = 1, 2, 3, \ldots.$$

H. Inequalities

$$(x>0,\ n = 1, 2, 3, \ldots)$$

$$\frac{n-1}{n}E_n(x) < E_{n+1}(x) < E_n(x)$$

$$\frac{1}{x+n} < e^x E_n(x) \leq \frac{1}{x+n-1}.$$

I. Related Functions

Incomplete Gamma Function

$\Gamma(1-n, x) = x^{1-n}E_n(x)$

Confluent Hypergeometric Function

$U(n, n, x) = e^x x^{1-n}E_n(x)$.

3. Method

Examination of the recurrence relation indicates that an independent computation must be carried out for at least one value of $n \geq 1$. The function $E_n(x)$ is always positive, for real x has no zeros, and is a slowly decreasing function with increasing n and/or x. Repeated application of the recurrence relation therefore will yield an increasing round off error even if, with more complicated scaling, both forward and backward recurrence relations are used starting at $n = [x]$. Generally, in physical applications the first few orders only are needed. Consequently, we have chosen an independent computing method valid for all orders.

The implementing American National Standard FORTRAN program has been set up in such a way as to require a minimal number of changes for varying precision, either single or double precision on the same or different computers. The program checks for positive arguments n (=RN) and x and integer n. If either is negative, an error indicator is set (IERR = 1) and an impossible value, the negative of the maximum machine value (=RINF), is returned for both functional values, $E_n(x)$ (=ENX) and $e^x E_n(x)$ (=EXPENX). If $n >$ RMAXI, the maximum integer convertible to a floating point number, n is assumed to be an integer. If $n \leq$ RMAXI, an integer test is applied to RN. If it fails the test, IERR is set equal to 2, and RINF is returned for both functional values. To assure ready portability, additional tests are performed for negative zero and allowance for round off errors due either to machine arithmetic or the system being used.

The special cases are treated independently. When $n = 0$, to avoid machine difficulty, x is tested against the reciprocal of the maximum machine value; if x equals, or is less than that number, the maximum value is returned for both functional values. If x is greater, then ENX = e^{-x}/x and EXPENX = $1/x$. When $x = 0$, and $n = 0$ or 1, ENX = EXPENX = RINF; for $n > 1$, ENX = EXPENX = $1/(n-1)$.

Computation with the series results in a greater round-off error in the neighborhood of $x = 1$, in particular for small values of n. Even if the form of the series were changed from the present alternating form, there is still a loss due to the logarithmic term. While the continued fraction is convergent for $x > 0$, the number of terms increases rapidly as x approaches 0. The integrand is not particularly amenable to automatic numerical integration by any of the low order formulas. In general ($n \leq$ RMAXI) the most accurate and efficient methods of automatic computation are the alternating power series for $x \leq 1$(ULPS) and the "even" form of the continued fraction for $x > 1$.

For $1 \leq n \leq$ RMAXI and $0 < x \leq 1$, the power series is used in the following form

$$E_n(x) = \text{ENX} = \sum_{M=0}^{K} T_M = \text{SUM} + TM \quad (M = 0, 1, \ldots, K)$$

where $T_M = \dfrac{-(-x)^M}{M!(M-n+1=D)} = \dfrac{\text{PTERM}}{D}$ for $M \neq n-1$

and $T_M = \text{PTERM}[\text{LOG}(x) - \text{PSI}(n)]$ for $M = n-1$.
PTERM is obtained by recurrence with PTERM(0) = −1 and PTERM$(M+1)$ = PTERM $(M) \cdot (-x)/M+1$.

Rather than compute a previously determined fixed number of terms, it is valid, provided $M \neq n-1$, to terminate the series when the relative error, computed as $|TM/\mathrm{SUM}|$ is less than or equal to a prescribed tolerance (=TOLER). In the present program, $\mathrm{TOLER} = 2^{-NBM}$, where NBM is the maximum number of binary digits in the mantissa of a floating point number. It may, however, be set to the accuracy desired.

Since the series is an alternating one, SUM may equal zero at some M. However, since the function has no zeros for real x, the relative error test may be safely bypassed if SUM = 0 and additional terms computed. In our manner of evaluating the series, TM may likewise be zero if $\log(x) = \Psi(n)$. (Values of $e^{\Psi(n)}$ for $n = 1(1)10$ are given in sec. 10.) The logarithmic term enters the computation for $n \leq 2$ since at least two terms of the series must be used with the above relative error test. Consequently, using the power series for $x \leq 1$, this situation will not then be critical when $n = 1$ and $x \approx 0.56$. However, since the code may be used for experimental purposes, the relative error test is not applied to the logarithmic term.

For $n \geq 1$ and $x > 1$, and $n > \mathrm{RMAXI}$, $x > 0$, the continued fraction in its "even" form

$$E_n(x) = e^{-x}\left[\frac{1}{x+n-}\,\frac{1\cdot n}{x+n+2-}\,\frac{2(n+1)}{x+n+4-}\cdots\right]$$

$$= e^{-x}F = e^{-x}\mathrm{EXPENX}$$

is evaluated in the forward direction. The first convergent $F_1/G_1 = A_1/B_1$ where $A_1 = 1$, $A_M = -(M-1)(n+M-2)$, $B_M = x+n+2(M-1)$. If we define

$$F_{-1} = 1,\ F_0 = 0,\ G_{-1} = 0 \text{ and } G_0 = 1$$

then successive convergents F_M/G_M for $M = 1, 2, \ldots$ may be obtained by the following recurrence relation

$$F_M = B_M F_{M-1} + A_M F_{M-2}$$

$$G_M = B_M G_{M-1} + A_M G_{M-2}.$$

Since in the basic continued fraction A_M and B_M are always positive with $A_M \geq 1$, F_M and G_M will always be positive and increasing with increasing M. Care must then be taken to ensure that overflow does not occur in generating the successive convergents. For the "even" form of the continued fraction, since $A_M < 0$ for $M > 1$, it is sufficient to check that G_M is always less than RINF/B_M. If G_M is greater, then F_M, F_{M-1}, G_M and G_{M-1} are all scaled by dividing by B_M.

Since, for the above continued fraction, the successive convergents form a monotonically increasing sequence, if through round-off errors,

$$1 - \frac{(F_{M-1}/G_{M-1})\ (=\mathrm{PREV})}{F_M/G_M} \leq 0 \qquad (F_M/G_M \leq F_{M-1}/G_{M-1})$$

then $F_{M-1}/G_{M-1} = F$.

If $0 < 1 - \dfrac{F_{M-1}/G_{M-1}}{F_M/G_M} \leq \mathrm{TOLER}\ (= 2^{-NBM})$, then $F_M/G_M = F$.

The following table gives an indication of the number of terms needed to obtain maximum machine accuracy for particular values of NBM, x and n with the various methods of computation.

Method	Number of Terms			
	NBM = 27		NBM = 60	
$x = 1$	$n = 1$	$n = 10$	$n = 1$	$n = 10$
Power Series	13	14	21	22
Continued Fraction	26	17	113	76
Numerical Integration (Trapezoidal or Simpson's Rule)	129	513	16385	65537
$x = 22$				
Asymptotic Expansion	22	.....	.....	.....
Continued Fraction	5	6	12	16
Numerical Integration	2049			
$x = 44$				
Asymptotic Expansion	10	35	45	.....
Continued Fraction	4	5	9	11
$x = 70$				
Asymptotic Expansion	8	15	23	62
Continued Fraction	4	5	8	10

4. Range

For the function $e^x E_n(x)$, the sum of x and n must be less than or equal to the maximum machine value. If the sum is greater, both $e^x E_n(x)$ and $E_n(x)$ are set equal to zero.

For the function $E_n(x)$ the range of x is dominant and essentially equivalent to the range for the exponential function. In single precision on the Univac 1108, $E_n(x) = 0$ beyond $x \approx 85$; in double precision beyond $x \approx 704$.

5. Accuracy

Using the Univac 1108 in computing $e^x E_n(x)$, we find the maximum relative error is 1.3(− 7) for the single precision computation and 2.4(− 17) for the double precision computation.

In computing $E_n(x)$, largely due to the error of the exponential routine, the maximum relative error is 4(− 7) for the single precision computation and 4.5(− 16) for the double precision computation.

The number of accurate binary digits is essentially the lesser of NBM−NBM$^{1/2}$ or NBM−I where I is the number of binary digits representing the integer part of x.

6. Precision

The precision may be set lower than the maximum by varying NBM or if desirable, deleting NBM and setting the proper value of TOLER.

7. Timing (Seconds—Univac 1108)

For $n=1, 2, 20$ over the range	NBM = 27	NBM = 60
$x=0(0.02)1$ (153 values)	0.18	0.46
$x=2(1)85$ (252 values)	.19	.61
$x=85(10)715$ (192 values)	–	.28
Maximum Time/Evaluation	.003	.015

8. Testing

The double precision results obtained were compared against available published values. Further check values were obtained by utilizing multi-precision packages. In addition, double precision check values were obtained, where appropriate, with the asymptotic expansion, numerical integration, and the "odd" form of the continued fraction. Single precision results were then checked against the double precision results. In all cases, the results obtained agreed within the reported accuracy.

9. Driver Program and Its Results

In the appendix we have included a driver program and its results. The usage of the subroutine is thus illustrated and a reasonable set of check values given to simplify the checkout of modifications to the subroutine. Two tables are given; one for the function $E_n(x)$ and the other for $e^x E_n(x)$. The functions are tabulated to 18 significant figures for $n=1$, 2, and 20, for $x=0$, $10^J(10^J)10^{J+1}$ with $J=$ JBEGIN(1) JEND. The value of JBEGIN has been set at -2; the value of JEND is the minimum value of J for which $E_1(x)=0$.

10. Special Constants

$E_1(1)=0.21938\ 39343\ 95520\ 27367\ 71637\ 75460\ 12164$

$\sum_{1}^{\infty}(-1)^{n+1}/n\cdot n!=E_1(1)+\gamma=0.79659\ 95992\ 97053\ 13428\ 36758\ 65542\ 52408$

$\Psi(1)=-\gamma$(Euler's constant)
$=-0.57721\ 56649\ 01532\ 86060\ 65120\ 90082\ 40243$
$\Psi(2)=1-\gamma$
$=0.42278\ 43350\ 98467\ 13939\ 34879\ 09917\ 59756$

$x=e^{\Psi(1)}=0.56145\ 94835\ 66885\ 16982\ 41432\ 14790\ 88078$
$e^{\Psi(2)}=1.52620\ 51115\ 95863\ 88047\ 48887\ 15036\ 77561$
$e^{\Psi(3)}=2.51628\ 68309\ 39363\ 58025\ 62371\ 75591\ 18353$
$e^{\Psi(4)}=3.51176\ 11663\ 39476\ 18136\ 68183\ 63142\ 18018$
$e^{\Psi(5)}=4.50919\ 05949\ 16874\ 93725\ 13380\ 93818\ 19453$
$e^{\Psi(6)}=5.50753\ 78297\ 01368\ 13780\ 61569\ 70031\ 98287$
$e^{\Psi(7)}=6.50638\ 71643\ 69172\ 08390\ 16850\ 25480\ 57156$
$e^{\Psi(8)}=7.50554\ 04760\ 51118\ 60659\ 92002\ 80578\ 56992$
$e^{\Psi(9)}=8.50489\ 15798\ 67776\ 22381\ 29291\ 76413\ 19981$
$e^{\Psi(10)}=9.50437\ 85180\ 84354\ 74744\ 37262\ 91500\ 55559$

$\log_e 2\ =0.69314\ 71805\ 59945\ 30941\ 72321\ 21458\ 17656$
$\log_e 10=2.30258\ 50929\ 94045\ 68401\ 79914\ 54684\ 36420$

$2^{-24} = 0.59604\ 64477\ 53906\ 25\ (-7)$
$2^{-27} = .74505\ 80596\ 92382\ 8125\ (-8)$
$2^{-36} = .14551\ 91522\ 83668\ 51806\ 64062\ 5\ (-10)$
$2^{-48} = .35527\ 13678\ 80050\ 09293\ 55621\ 33789\ 0625\ (-14)$
$2^{-60} = .86736\ 17379\ 88403\ 54720\ 59622\ 40695\ 95336\ (-18)$
$2^{-108} = .30814\ 87911\ 01957\ 73648\ 89564\ 70813\ 58837\ (-32)$

Maximum and Minimum Machine Values and Their Natural Logarithms

NBC = Number of binary digits in the (biased) characteristic of a floating point number

$$2^{-(2^{NBC-1}+1)} \leq x < 2^{2^{NBC-1}-1}$$

NBC = 8

$2^{127} = 0.17014\ 11834\ 60469\ 23173\ 16873\ 03715\ 88410\ (39)$
$2^{-129} = 0.14693\ 67938\ 52785\ 93849\ 60920\ 67152\ 78070\ (-38)$
$\log_e(2^{127}) = 88.02969\ 19311\ 13054\ 29598\ 84794\ 25188\ 42414$
$\log_e(2^{-129}) = -89.41598\ 62922\ 32944\ 91482\ 29436\ 68104\ 77728$

NBC = 11

$2^{1023} = 0.89884\ 65674\ 31157\ 95386\ 46525\ 95394\ 51236\ (308)$
$2^{-1025} = 0.27813\ 42323\ 13400\ 17288\ 62790\ 89666\ 55050\ (-308)$
$\log_e(2^{1023}) = 709.08956\ 57128\ 24051\ 53382\ 84602\ 51714\ 62914$
$\log_e(2^{-1025}) = -710.47586\ 00739\ 43942\ 15266\ 29244\ 94630\ 98227$

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Appendix

Implementing Program

```
      SUBROUTINE EXPINT (RN,X,ENX,EXPENX,IERR)
C
C LANGUAGE.  AMERICAN NATIONAL STANDARD FORTRAN
C
C DEFINITIONS.
C   EN(X)= INTEGRAL (EXP(-X*T)DT/(T**N)), FROM 1 TO INFINITY
C             RN(=N), POSITIVE INTEGER
C                  X, REAL AND POSITIVE
C
C  SPECIAL CASES
C   EN(0)=INFINITY(=RINF, MAXIMUM MACHINE VALUE)    N .LE. 1
C   EN(0)= 1/N-1      N .GT. 1
C
C   E0(X)= EXP(-X)/X     X .GT. 1/RINF
C   E0(X)= INFINITY      X .LE. 1/RINF
C
C USAGE.      CALL EXPINT (RN,X,ENX,EXPENX,IERR)
C
C  FORMAL PARAMETERS
C   RN  REAL OR DOUBLE PRECISION TYPE                    INPUT
C       FOR A POSITIVE INTEGER N
C   X                          (SAME TYPE AS RN)         INPUT
C   ENX= EN(X)                  (SAME TYPE AS X)        OUTPUT
C   EXPENX= EXP(X)*EN(X)        (SAME TYPE AS X)        OUTPUT
C   IERR INTEGER VARIABLE                               OUTPUT
C         NORMAL RETURN  IERR= 0
C         ERROR RETURN   IERR= 1, X AND/OR N NEGATIVE
C                    ENX=EXPENX=-INFINITY (IMPOSSIBLE VALUE)
C                        IERR= 2, N NON-INTEGER
C                    ENX=EXPENX=INFINITY
C
C  MODIFICATIONS
C   DOUBLE PRECISION UNIVAC 1108 RESULTS ARE OBTAINED IF AS
C   SET UP BELOW WHERE
C
C     NBM=ACCURACY DESIRED OR MAXIMUM NUMBER OF BINARY
C         DIGITS IN THE MANTISSA OF A FLOATING POINT NUMBER
C
C   WITH
C    (1) THE DOUBLE PRECISION TYPE STATEMENT
C    (2) THE MAXIMUM MACHINE VALUE AND THE MAXIMUM INTEGER
C        (=RMAXI) CONVERTIBLE TO A FLOATING POINT NUMBER
C        GIVEN AS DOUBLE PRECISION CONSTANTS
C    (3) DOUBLE PRECISION DECIMAL CONSTANTS
C    (4) DATA STATEMENT NBM=60 FOR THE CONTROL VARIABLE
C    (5) FUNCTION TYPE STATEMENTS - DEXP, DLOG.
C
C   SINGLE PRECISION UNIVAC 1108 RESULTS ARE OBTAINED BY
C      (1) DELETING THE DOUBLE PRECISION TYPE STATEMENT
C      (2) ADJUSTING MAXIMUM MACHINE VALUE
C      (3) CHANGING THE D'S TO E'S ON THE DATA CARDS FOR ALL
C          DECIMAL CONSTANTS
C      (4) SETTING NBM=27 IN THE CONTROL VARIABLE DATA
C      (5) CHANGING FUNCTION TYPE - EXP, LOG.
C
```

```
      C     FOR OTHER COMPUTERS THE APPROPRIATE VALUE OF NBM MUST
      C     BE INSERTED AND ALL VALUES ADJUSTED ACCORDINGLY.
      C
      C   IF A PRECOMPUTED VALUE OF TOLER(=2**(-NBM)) IS
      C   INCLUDED IN A DATA STATEMENT, COMPUTATION OF THE CONTROL
      C   VARIABLE MAY BE OMITTED AND THE DATA STATEMENT FOR NBM
      C   DELETED.
      C
      C   CAUTION - THE SUBROUTINE CANNOT READILY BE ADAPTED TO
      C             COMPUTE THE EXPONENTIAL INTEGRAL FOR A COMPLEX
      C             ARGUMENT AS THE CONTINUED FRACTION IS INVALID
      C             ALONG THE NEGATIVE REAL AXIS. IN ADDITION MANY
      C             OF THE COMPARISONS BECOME MEANINGLESS.
      C
      C METHOD.
      C  POWER SERIES,     X .LE. 1(=ULPS, UPPER LIMIT FOR POWER
      C                                    SERIES)
      C    ENX = SUM(TM),    M=0,1,2,...,K
      C     TM =-((-X)**M/1*2*3...M)/(M-N+1=D)       M .NE. N-1
      C        = PTERM/D
      C     TM = PTERM*(LOG(X)-PSI(N))     M .EQ. N-1
      C      PTERM(0)=-1
      C      PTERM(M+1)= PTERM(M)*(-X)/(M+1)
      C      PSI(N)= -EULER+1+1/2+ ... +1/(N-1)
      C      IF R.E.(=ABS(TM/SUM)) .LE. TOLER (=2**(-NBM)),      M=K
      C
      C  CONTINUED FRACTION,     X .GT. 1
      C   EN(X)=EXP(-X)*(1I/I(X+N)-    1*N I/I(X+N+2)-
      C                              2*(N+1)I/I(X+N+4)-...
      C   EN(X)=EXP(-X)*II(AM I/I BM)          M=1,2,...,K
      C     AM(1)=1     AM(M)=-(M-1)*(N+M-2)
      C     BM(M)=X+N+2*(M-1)
      C
      C   EN(X)=EXP(-X)*FM(K)/GM(K)=EXP(-X)*F(K)
      C        IF R.E. (=ABS(1-PREV/F)) .LE. TOLER(=2**(-NBM)),M=K
      C     F=FM/GM      PREV=FMM1/GMM1
      C
      C RANGE.
      C  FOR EXP(X)*EN(X)     (N+X) .LE. MAXIMUM MACHINE VALUE
      C  FOR EN(X)    X=APPROXIMATELY 85.0, SINGLE PRECISION
      C                               704.0, DOUBLE PRECISION
      C  BEYOND THIS RANGE EN(X)=0
      C
      C ACCURACY.  THE NUMBER OF ACCURATE BINARY DIGITS IS ESSEN-
      C            TIALLY THE LESSER OF
      C                NBM - SQUARE ROOT OF NBM
      C                NBM - I(NUMBER OF BINARY DIGITS REPRESENT-
      C                      ING THE INTEGER PART OF X)
      C                      (ACCURACY OF THE EXPONENTIAL ROUTINE)
      C
      C PRECISION. VARIABLE - BY SETTING THE DESIRED NBM OR TOLER.
      C
      C MAXIMUM TIME.                UNIVAC 1108-EXEC II
      C    (SECONDS)                  NBM=27    NBM=60
      C                                .0025     .015
      C
      C STORAGE.   COMPILED BY THE UNIVAC 1108, EXEC 8/FORTRAN V,
      C            THIS SUBROUTINE REQUIRES 478 WORDS OF STORAGE.
      C
      C
      C
```

```
117       C
118       C
119       C
120       C
121       C
122       C              MACHINE DEPENDENT STATEMENTS
123       C
124             DOUBLE PRECISION AM,ASUM,BM,D,ENX,EULER,
125            1        EXPENX,EXPNX,F,FM,FMM1,FMM2,GM,GMM1,
126            2        GMM2,HALF,ONE,ONPTEV,PREV,PSI,PTERM,
127            3        RINF,RM,RMAXI,RN,RNM1,RRN,SUM,TEMP,
128            4        TM,TOLER,TWO,ULPS,X,XLOG,ZERO
129       C
130       C                    CONSTANTS
131       C                      MAXIMUM MACHINE VALUE
132       C                      MAXIMUM CONVERTIBLE INTEGER
133       C
134             DATA RINF / .898846567431157954D308 /
135             DATA RMAXI / 134217727.D0 /
136       C
137       C                    ADDITIONAL CONSTANTS
138       C
139             DATA EULER / .57721566490153286D0 /
140             DATA HALF,ONE,ONPTEV,TWO,ULPS,ZERO /
141            1   .5D0,1.D0,1.5D0,2.D0,1.D0,0.0D0 /
142       C
143       C                    CONTROL VARIABLE
144       C
145       C     NBM=ACCURACY DESIRED OR MAXIMUM NUMBER OF BINARY
146       C          DIGITS IN THE MANTISSA OF A FLOATING POINT NUMBER
147       C
148             DATA NBM / 60 /
149             TOLER=TWO**(-NBM)
150       C
151       C           VALIDITY TEST FOR INPUT PARAMETERS
152       C           ERROR CONDITION, IMPOSSIBLE VALUES RETURNED
153       C
154       C               NEGATIVE ZERO CHECKS
155       C
156             IERR=0
157             IF (RN .LT. ZERO) GO TO 3
158             IF (RN .GT. RMAXI) GO TO 10
159       C
160       C                   VALIDITY TEST FOR N INTEGER
161       C
162             N=RN
163             RRN=N
164             IF (RRN) 2,1,2
165          1  IF ((RN-RRN)-TOLER) 10,10,2
166          2  IF ((ONE-(RRN/RN)) .LE. TOLER) GO TO 10
167             RRN=RRN+ONE
168             IF ((ONE-(RN/RRN)) .LE. TOLER) GO TO 10
169       C
170       C                    ERROR RETURN
171       C                    N NON-INTEGER
172       C
173             IERR=2
174             ENX=RINF
175             EXPENX=ENX
176             RETURN
177       C
```

```
178          3  IF (-RN) 12,10,12
179       C
180         10  IF (X) 11,30,20
181         11  IF (-X) 12,30,12
182       C
183       C                   ERROR RETURN
184       C                   X AND/OR N NEGATIVE
185       C
186         12  IERR=1
187             ENX=-RINF
188             EXPENX=ENX
189             RETURN
190       C
191       C             FUNCTION TYPE STATEMENTS
192       C
193         20  EXPNX=DEXP(-X)
194             XLOG=DLOG(X)
195       C
196             IF (RN .GE. HALF) GO TO 60
197       C
198       C             SPECIAL CASES
199       C
200             IF (X .LE. ONE/RINF) GO TO 40
201             EXPENX=ONE/X
202             ENX=EXPNX*EXPENX
203             RETURN
204       C
205         30  IF (RN .LT. ONPTFV) GO TO 40
206             ENX=ONE/(RN-ONE)
207             GO TO 50
208       C
209         40  ENX=RINF
210         50  EXPENX=ENX
211             RETURN
212       C
213         60  IF (RN .GT. RMAXI) GO TO 70
214             IF (X .LE. ULPS) GO TO 100
215         70  IF (X .LE. (RINF-RN)) GO TO 200
216             EXPENX=ZERO
217             ENX=EXPENX
218             RETURN
219       C
220       C        METHOD --- POWER SERIES
221       C
222        100  RM=0
223             PTERM=-ONE
224             SUM=0
225             PSI=-EULER
226             D=-(RN-ONE)
227       C
228        110  IF (D .GE. HALF) GO TO 130
229               IF (-D .GE. HALF) GO TO 120
230       C
231       C             COMPUTE TM FOR M .EQ. N-1
232       C
233               SUM=PTERM*(XLOG-PSI)+SUM
234               GO TO 170
235       C
236       C             COMPUTE PSI(N)
237       C
238        120    PSI=PSI+(ONE/(RM+ONE))
```

```
C
C              COMPUTE TM FOR M .NE. N-1
C
 130     TM=PTERM/D
         SUM=TM+SUM
C
C              TOLERANCE CHECK
C
C                   ZERO CHECKS
C
         IF (SUM .LT. ZERO) GO TO 140
         ASUM=SUM
         GO TO 150
C
 140     ASUM=-SUM
 150     IF (ASUM) 160,170,160
 160     IF (TM .LT. ZERO) TM=-TM
         IF ( TM/ASUM .GT. TOLER) GO TO 170
C
       ENX= SUM
       EXPENX=ENX/EXPNX
       RETURN
C
C              ADDITIONAL TERMS
C
 170     RM=RM+ONE
         D=D+ONE
         PTERM=-(X*PTERM)/RM
         GO TO 110
C
C          METHOD --- CONTINUED FRACTION
C
 200   RM=ONE
       FMM2=ONE
       GMM2=0
       FMM1=0
       GMM1=ONE
       PREV=FMM1/GMM1
       AM=ONE
       BM=X+RN
       RNM1=RN-ONE
C
 210   FM=BM*FMM1 + AM*FMM2
         GM=BM*GMM1 + AM*GMM2
         F=FM/GM
C
C              TOLERANCE CHECK.
C
         TEMP=ONE-(PREV/F)
         IF (TEMP .GT. ZERO) GO TO 220
C
       EXPENX=PREV
       GO TO 230
C
 220     IF (TEMP .GT. TOLER) GO TO 240
C
       EXPENX=F
 230   ENX=EXPNX*EXPENX
       RETURN
C
 240     IF( GM .LT. RINF/BM) GO TO 250
```

```
300       C
301       C              SCALING
302       C
303       C BOTH FM AND GM MUST BE TESTED IF N=1 AND X .LT. .44
304       C SCALING SHOULD NOT BE DELETED AS THE VALUES OF FM AND GM
305       C MAY OVERFLOW FOR SOME CHOICES OF THE PARAMETERS.
306       C
307                    FMM1=FMM1/BM
308                    GMM1=GMM1/BM
309                    FM=FM/BM
310                    GM=GM/BM
311       C
312       C              ADDITIONAL CONVERGENTS
313       C
314        250       AM=-RM*(RNM1+RM)
315                  RM=RM+ONE
316                  BM=BM+TWO
317                  FMM2=FMM1
318                  GMM2=GMM1
319                  FMM1=FM
320                  GMM1=GM
321                  PREV=F
322                  GO TO 210
323       C
324             END
```

Driver (Test) Program

```
      C
      C DOUBLE PRECISION TEST PROGRAM
      C ALGORITHM FOR EXPONENTIAL INTEGRAL EN(X) AND EXP(X)*EN(X)
      C RN(=N,A POSITIVE INTEGER), X, REAL AND POSITIVE
      C (AMERICAN NATIONAL STANDARD FORTRAN - UNIVAC 1108)
      C
      C USAGE OF CALL EXPINT (RN,X,ENX,EXPENX,IERR)
      C
      C      TABLE OF ENX(X(L),N(K)) AND EXPENX(X(L),N(K))
      C                  JT=1                  JT=2
      C               L=1,2,...,NCX,  K=1,2,...,NN
      C  N=1,2,20    X=0, 10**J (10**J) 10**(J+1)    J=JBEGIN(1)JEND
      C                  (10**J)(ID=1,2,...,9)          JBEGIN=-2
      C  JEND=MIN J FOR ENX(X=(10**J)*ID,N(1))=0
      C
      C   NLINE = NUMBER OF AVAILABLE PRINT LINES PER PAGE
      C   NFCOL = NUMBER OF FUNCTIONAL COLUMNS PER PAGE
      C   LINHD = NUMBER OF HEADING LINES PER PAGE
      C
      C   RMAXI = MAXIMUM CONVERTIBLE INTEGER
      C
      C  DIMENSION N(NN),X(NX),ENX(NX,NN),EXPENX(NX,NN)
      C
      C            MACHINE DEPENDENT STATEMENTS
      C            INPUT DATA
      C
            DOUBLE PRECISION D,DFCTR,DX,ENX,ETPTFV,EXPENX,
           1                 PTFIVE,RMAXI,RN,TEN,X,ZERO
      C
            DIMENSION N(3),X(100),ENX(100,3),EXPENX(100,3)
      C
            DATA RMAXI / 134217727.D0 /
            DATA ETPTFV,PTFIVE,ZERO / 8.500,.5D0,0.0D0 /
      C
            DATA NN,NX,NLINE,NFCOL,LINHD/ 3,100,58,3,3 /
            DATA N(1),N(2),N(3) / 1,2,20 /
            DATA JBEGIN / -2 /
      C
      1000  FORMAT (1H1,38X,5HEN(X)//)
      1001  FORMAT (4X,1HX,9X,2HN=,I12,4(13X,I12))
      1002  FORMAT (1D7.1,5D25.18)
      1003  FORMAT (1H1,35X,12HEXP(X)*EN(X)//)
      C
      C            FUNCTION TYPE STATEMENT
      C
            MAXJI=DLOG10(RMAXI)
      C
      C              COMPUTATION AND STORAGE OF FUNCTIONS
      C
      C                SET UP INITIAL X
      C
            D=ZERO
      C
      C                SET UP INITIAL DECADE
      C
            TEN=10
            L=1
            J=JBEGIN
            ID=1
```

```
      C
            IF (IABS(J) .GT. MAXJT) GO TO 1
            DFCTR=10**(IABS(J))
            GO TO 2
      C
         1  DFCTR=TEN**(IABS(J))
      C
         2  IF (J .LE. 0) GO TO 3
              DX=D*DFCTR
              GO TO 4
      C
         3    DX=D/DFCTR
      C
         4    X(L)=DX
      C
              DO 5 K=1,NN
                RN=N(K)
         5      CALL EXPINT(RN,DX,ENX(L,K),EXPFNX(L,K),IERR)
      C
              IF (ENX(L,1) .LE. ZERO) GO TO 9
              IF (L .GE. NX) GO TO 9
              IF (D .LT. PTFIVE) GO TO 8
              IF (D .GT. ETPTFV) GO TO 6
              ID=ID+1
              GO TO 8
      C
         6    J=J+1
              ID=1
              IF (J .LE. 0) GO TO 7
              DFCTR=DFCTR*TEN
              GO TO 8
      C
         7    DFCTR=DFCTR/TEN
      C
         8    D=ID
              L=L+1
              GO TO 2
      C
         9  NCX=L
      C
      C                PRINTING OF TABLES
      C                L=1,2,...,NCX
      C                K=1,2,...,NN
      C
            JT=1
      C
        21  NA=1
              NB=NFCOL
      C
        22    IF (NB .GT. NN) NB=NN
                L=1
                IF (X(1)) 24,23,24
        23      NPLINE=1
                LCOUNT=NLINE-LINHD-1
                INDEXS=9
                GO TO 26
      C
        24      NPLINE=0
      C
        25      LCOUNT=NLINE-LINHD
                  INDEXS=1
```


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```
121          26         NPLINE=NPLINE+9*(LCOUNT/10)
122                     IF (JT .EQ. 1) GO TO 27
123                     WRITE (6,1003)
124                     GO TO 28
125          C
126            27       WRITE (6,1000)
127            28       WRITE (6,1001) (N(K),K=NA,NB)
128          C
129            29       IF (JT .EQ. 1) GO TO 30
130                       WRITE (6,1002) X(L),(EXPENX(L,K),K=NA,NB)
131                       GO TO 31
132          C
133            30         WRITE (6,1002) X(L),(ENX(L,K),K=NA,NB)
134            31         IF (L .GE. NCX) GO TO 35
135                       IF (L .GE. NPLINE) GO TO 34
136                       IF (INDEXS .EQ. 9) GO TO 32
137                       INDEXS=INDEXS+1
138                       GO TO 33
139          C
140            32         WRITE (6,1004)
141                       INDEXS=1
142          C
143            33         L=L+1
144                       GO TO 29
145          C
146          C                      EXCESS X'S
147          C
148            34       L=L+1
149                     GO TO 25
150          C
151          C                    EXCESS N'S
152          C
153            35     IF (NB .GE. NN) GO TO 36
154                   NA=NA+NFCOL
155                   NB=NB+NFCOL
156                   GO TO 22
157          C
158          C                 ADDITIONAL FUNCTION
159          C
160            36   IF (JT .GE. 2) STOP
161                 JT=JT+1
162                 GO TO 21
163          C
164          1004  FORMAT (1H )
165          C
166                END
```

Test Results

EN(X)

X	N= 1	2	20
.0	.89884656743115795 1+308	.10000000000000000+001	.52631578947368421 0-001
.1-001	.40379295765381138 4+001	.94967053798378691 6+000	.52078954179335147 6-001
.2-001	.33547077833097095 2+001	.91310451764056111 2+000	.51532149651352828 5-001
.3-001	.29591187240212806 9+001	.88167197182786975 8+000	.50991103854550287 2-001
.4-001	.26812636890252799 2+001	.85353889159131201 4+000	.50455755932602586 5-001
.5-001	.24678984885099743 7+001	.82783450007521529 2+000	.49926045674777729 8-001
.6-001	.22953069181437823 3+001	.80404611849562177 0+000	.49401913509057826 9-001
.7-001	.21508381802567988 5+001	.78183514728797231 0+000	.48883300495333925 2-001
.8-001	.20269410025857414 8+001	.76096106617977646 6+000	.48370148318673710 0-001
.9-001	.19187447700326628 6+001	.74124415596828853 0+000	.47862399282661289 3-001
.1+000	.18229239584193906 7+001	.72254502219402050 9+000	.47359996302808289 7-001
.2+000	.12226505441838930 9+001	.57420064424120324 5+000	.42617865691013849 6-001
.3+000	.90567665167584671 4+000	.46911522517896385 4+000	.38351849086885191 7-001
.4+000	.70238011886566248 0+000	.38936799489374312 4+000	.34514000209804862 3-001
.5+000	.55977359477616081 0+000	.32664386232455301 7+000	.31061217393630982 3-001
.6+000	.45437950318940211 1+000	.27618393418038516 9+000	.27954752219493567 1-001
.7+000	.37376884323350914 4+000	.23494711352795311 4+000	.25159768199725304 1-001
.8+000	.31059657854554303 5+000	.20085170128078716 5+000	.22644944389666645 6-001
.9+000	.26018393932599964 0+000	.17240411434719943 7+000	.20382119331468755 3-001
.1+001	.21938393439552027 3+000	.14849550677592204 8+000	.18345971206755873 3-001
.2+001	.48900510708061120 5-001	.37534261820490453 0-001	.64143058553248993 7-002
.3+001	.13048381094197037 6-001	.10641925085272830 6-001	.22486480641400091 9-002
.4+001	.37793524098489064 6-002	.31982292493385543 7-002	.79018909980371519 2-003
.5+001	.11482955912753258 1-002	.99646904270883811 8-003	.27827459288573081 4-003
.6+001	.36008245216265865 6-003	.31825746369040646 5-003	.98188422730249018 7-004
.7+001	.11548173161033821 6-003	.10350984428214869 3-003	.34706848624884370 7-004
.8+001	.37665622843924901 8-004	.34137645151112624 7-004	.12287754356666306 1-004
.9+001	.12447354178006272 1-004	.11383616484623100 4-004	.43568742044296172 5-005
.1+002	.41569689296853242 7-005	.38302404656316087 5-005	.15469362798777248 5-005
.2+002	.98355252906498816 8-010	.94048564308581489 7-010	.52164814650796305 3-010
.3+002	.30215520106888125 3-014	.29296693677373696 9-014	.18862597517156085 4-014
.4+002	.10367732614516569 7-018	.10126120948496110 7-018	.71192823207980889 4-019
.5+002	.37832640295504590 1-023	.37117833186688273 6-023	.27664230809768887 1-023
.6+002	.14358675656812567 9-027	.14130536608979607 0-027	.10979319276125777 8-027
.7+002	.56003062858134384 4-032	.55235335839239895 7-032	.44279144061699196 8-032
.8+002	.22285432586884729 0-036	.22016780894636843 3-036	.18084114736535707 0-036
.9+002	.90054740588674110 3-041	.89085971009845496 3-041	.74612549899422477 8-041
.1+003	.36835977616820321 7-045	.36478214338803782 6-045	.31043160395160962 5-045
.2+003	.68852261063076355 9-089	.68513054752104111 0-089	.62930179325203699 7-089
.3+003	.17103842768045101 1-132	.17047391998483434 0-132	.16091250255379243 3-132
.4+003	.47760135864209722 2-176	.47641621456168031 2-176	.45604423385382053 7-176
.5+003	.14220767822536384 2-219	.14192495473093421 0-219	.13702118216839136 4-219
.6+003	.44099897945098379 6-263	.44026762984080333 5-263	.42750548862182149 4-263
.7+003	.14065187662340329 2-306	.14045180121540397 2-306	.13694522116512558 9-306
.8+003	.00000000000000000	.00000000000000000	.00000000000000000

EXP(X)*EN(X)

X	N= 1	2	20
.0	.898846567431157951+308	.10000000000000000+001	.526315789473684210-001
.1-001	.407851144345642585+001	.959214885565435741+000	.526023563704061981-001
.2-001	.342247737593075321+001	.931550452481384936+000	.525731681287694728-001
.3-001	.304923730567447425+001	.908522880829765773+000	.525440141582159931-001
.4-001	.279068813598834047+001	.888372474560466383+000	.525148943946742833-001
.5-001	.259443034976061332+001	.870278482511969335+000	.524858087742430369-001
.6-001	.243724077122346628+001	.853765553726592023+000	.524567572331905115-001
.7-001	.230679154487934795+001	.838524591858445643+000	.524277397079539275-001
.8-001	.219575897504124847+001	.824339281996700124+000	.523987561351388700-001
.9-001	.209944118436360746+001	.811050293407275329+000	.523698064515186901-001
.1+000	.201464254470845168+001	.798535745529154834+000	.523408905940339148-001
.2+000	.149334874693223961+001	.701330250613552081+000	.520535787019039881-001
.3+000	.122253560508058556+001	.633239318475824335+000	.517695812767576684-001
.4+000	.104782800845600643+001	.580868796617597430+000	.514888379273828789-001
.5+000	.922910632483730466+000	.538544683758134765+000	.512112898107201955-001
.6+000	.827933435273508820+000	.503239938835894713+000	.509368795793975367-001
.7+000	.752678020029587137+000	.473125385979289004+000	.506655513315264279-001
.8+000	.691245397802831489+000	.447003681757734811+000	.503972505626390015-001
.9+000	.639949226639299739+000	.424045696024630237+000	.501319241196527574-001
.1+001	.596347362323194074+000	.403652637676805926+000	.498695201567573519-001
.2+001	.361328616888222592+000	.277342766223554833+000	.473956658006950287-001
.3+001	.262083740255318501+000	.213748779234044509+000	.451653037195386256-001
.4+001	.206345649901055832+000	.174617400395776667+000	.431428630256384291-001
.5+001	.170422176284732204+000	.147889118576338992+000	.412996114281546988-001
.6+001	.145267629233886893+000	.128394224596678636+000	.396120369170190107-001
.7+001	.126641096076632765+000	.113512327463570647+000	.380606810266053526-001
.8+001	.112279639253499312+000	.101762885972005506+000	.366292794923112188-001
.9+001	.100861955580640929+000	.922423997742316384-001	.353041176322005723-001
.1+002	.915633339397880818-001	.843666606021191810-001	.340735390554722559-001
.2+002	.477185454959608417-001	.456290900807831660-001	.253085524935690803-001
.3+002	.322897387589801252-001	.313078372305962434-001	.201574668908614936-001
.4+002	.244041150796285762-001	.238353968148569492-001	.167577416876913709-001
.5+002	.196151099301148704-001	.192445034942564817-001	.143430890423517574-001
.6+002	.163977137080465268-001	.161371775172083932-001	.125384637484814277-001
.7+002	.140872270003268122-001	.138941099771231436-001	.111381471287998954-001
.8+002	.123475166636786097-001	.121986669057112264-001	.100197250911190282-001
.9+002	.109903102083356454-001	.108720812497919106-001	.910574016947094732-002
.1+003	.990194228673301838-002	.980577132669815934-002	.834476515943799403-002
.2+003	.497524632317935662-002	.495073536412867515-002	.454731825027371299-002
.3+003	.332229556527070706-002	.331133041878788067-002	.312560692284812426-002
.4+003	.249378101793988503-002	.248759282404598474-002	.238122114425943504-002
.5+003	.199601590476041089-002	.199204761979455498-002	.192321864972550092-002
.6+003	.166389810215794723-002	.166113870523165916-002	.161298688732910590-002
.7+003	.142653641830088669-002	.142450718937931575-002	.138894233048444955-002
.8+003	.124844139167435033-002	.124688666051973815-002	.121954838359794140-002

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